

MEASURABILITY IN $C(2^\kappa)$ AND KUNEN CARDINALS

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ABSTRACT. A cardinal κ is called a Kunen cardinal if the σ -algebra on $\kappa \times \kappa$ generated by all products $A \times B$, where $A, B \subset \kappa$, coincides with the power set of $\kappa \times \kappa$. For any cardinal κ , let $C(2^\kappa)$ be the Banach space of all continuous real-valued functions on the Cantor cube 2^κ . We prove that κ is a Kunen cardinal if and only if the Baire σ -algebra on $C(2^\kappa)$ for the pointwise convergence topology coincides with the Borel σ -algebra on $C(2^\kappa)$ for the norm topology. Some other links between Kunen cardinals and measurability in Banach spaces are also given.

1. INTRODUCTION

In every completely regular topological space T there are two natural σ -algebras: the Borel σ -algebra $\text{Bo}(T)$ generated by all open sets and, usually much smaller, the Baire σ -algebra $\text{Ba}(T)$ generated by all continuous real-valued functions on T . For a Banach space X , we always have

$$\text{Ba}(X_w) \subset \text{Bo}(X_w) \subset \text{Bo}(X) = \text{Ba}(X)$$

where X_w stands for X equipped with its weak topology. Moreover, for the Banach space $C(K)$ of all continuous real-valued functions on a compact space K , other σ -algebras appear:

$$\begin{array}{ccc} \text{Ba}(C_p(K)) & \subset & \text{Bo}(C_p(K)) \\ \cap & & \cap \\ \text{Ba}(C_w(K)) & \subset & \text{Bo}(C_w(K)) \subset \text{Bo}(C(K)) \end{array}$$

where $C_p(K)$ (resp. $C_w(K)$) stands for $C(K)$ equipped with the pointwise convergence (resp. weak) topology. It is well-known that all these σ -algebras coincide for separable Banach spaces. For nonseparable Banach spaces some of the inclusions above might be strict and the equalities between these σ -algebras are closely related to several interesting properties of X and K , see e.g. [2, 3, 9, 10, 18, 19, 26].

The first example of a nonseparable Banach space X for which $\text{Ba}(X_w) = \text{Bo}(X)$ was given by Fremlin [12] showing that such equality holds for $X = \ell^1(\omega_1)$. For

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any cardinal κ , Fremlin proved that the equality

$$\text{Ba}(\ell^1(\kappa)_w) = \text{Bo}(\ell^1(\kappa))$$

is equivalent to saying that

$$(1.1) \quad \mathcal{P}(\kappa \times \kappa) = \mathcal{P}(\kappa) \otimes \mathcal{P}(\kappa)$$

(i.e. the power set of $\kappa \times \kappa$ coincides with the σ -algebra on $\kappa \times \kappa$ generated by all products $A \times B$, where $A, B \subset \kappa$). From now on we shall say that a cardinal κ is a *Kunen cardinal* if (1.1) holds. This notion has its origin in a problem posed by Ulam [30] and was investigated by Kunen in his doctoral dissertation [17]. Let us mention that:

- (i) any Kunen cardinal is less than or equal to \mathfrak{c} ;
- (ii) ω_1 is a Kunen cardinal;
- (iii) \mathfrak{c} is a Kunen cardinal under Martin's axiom, while it is relatively consistent that \mathfrak{c} is not a Kunen cardinal.

Kunen cardinals have been also considered by Talagrand [27] in connection with measurability properties of Banach spaces, and in a paper by Todorćević [29] on universality properties of ℓ_∞/c_0 , where the reader can find more accurate historical remarks on this topic.

In this paper we focus on the Banach space $C(2^\kappa)$ for a cardinal κ and prove that the equality

$$\text{Ba}(C_p(2^\kappa)) = \text{Bo}(C(2^\kappa))$$

holds if and only if κ is a Kunen cardinal (Theorem 2.8). This extends Fremlin's aforementioned result, since $C(2^\kappa)$ contains $\ell^1(\kappa)$ isomorphically. The picture of coincidence of σ -algebras on $C(2^\kappa)$ is then the following:

- (a) $\text{Bo}(C_p(2^\kappa)) = \text{Bo}(C(2^\kappa))$ for any κ , since $C(2^\kappa)$ admits a pointwise Kadec equivalent norm, see e.g. [4, VII.1.10] and [10].
- (b) $\text{Ba}(C_p(2^\kappa)) = \text{Ba}(C_w(2^\kappa))$ if and only if $\kappa \leq \mathfrak{c}$. Indeed, the “if” follows from the fact that any Radon probability on $2^\mathfrak{c}$ admits a uniformly distributed sequence (cf. [13, 491Q]). On the other hand, if $\kappa > \mathfrak{c}$ then 2^κ is nonseparable and so the standard product measure on 2^κ cannot be $\text{Ba}(C_p(2^\kappa))$ -measurable (cf. [25, Proposition 3.6]).
- (c) $\text{Ba}(C_p(2^\kappa)) = \text{Bo}(C(2^\kappa))$ if and only if κ is a Kunen cardinal.

The paper is organized as follows. Section 2 is entirely devoted to prove statement (c) (Theorem 2.8). The proof is self-contained and rather technical.

In Section 3 we single out a certain topological property of a compact space K which guarantees that $\text{Ba}(C_p(K)) = \text{Bo}(C_p(K))$ (Corollary 3.4). That property holds for $K = 2^{\omega_1}$ and this gives a more direct proof of the equality $\text{Ba}(C_p(2^{\omega_1})) = \text{Bo}(C(2^{\omega_1}))$ which relies on statement (a) above.

In Section 4 we show that a Banach space X admits a non $\text{Ba}(X_w)$ -measurable equivalent norm whenever X has a biorthogonal system of non Kunen cardinality (Theorem 4.4): this applies to $C(2^\kappa)$ and $\ell^1(\kappa)$ provided that κ is not Kunen.

Terminology. For any $n \in \mathbb{N}$ we write $2^n := \{0, 1\}^n$. As usual, ω_1 denotes the first uncountable ordinal and \mathfrak{c} is the cardinality of the continuum. All our topological spaces are assumed to be Hausdorff. Given a measurable space (Y, Σ) and $S \subset Y$, the *trace of Σ on S* is the σ -algebra on S defined by $\{S \cap A : A \in \Sigma\}$.

Given any set Γ , we write $\mathcal{P}(\Gamma)$ to denote the power set of Γ . The symbol $|\Gamma|$ stands for the cardinality of Γ . The σ -algebra on $\Gamma^2 = \Gamma \times \Gamma$ generated by all products $A \times B$, where $A, B \subset \Gamma$, is denoted by $\mathcal{P}(\Gamma) \otimes \mathcal{P}(\Gamma)$. For any $U \subset \Gamma$, the characteristic function $1_U : \Gamma \rightarrow \{0, 1\}$ is defined by $1_U(\gamma) = 1$ if $\gamma \in U$, $1_U(\gamma) = 0$ if $\gamma \notin U$. We denote by 2^Γ the Cantor cube, i.e. the set of all $\{0, 1\}$ -valued functions on Γ , which becomes a compact space when equipped with the pointwise convergence topology. $\mathcal{P}(\Gamma)$ and 2^Γ can be identified via $U \mapsto 1_U$.

Given a set E and $\mathcal{F} \subset \mathbb{R}^E$, we write $\sigma(\mathcal{F})$ to denote the σ -algebra on E generated by \mathcal{F} (i.e. the smallest one for which every $f \in \mathcal{F}$ is measurable). It is well-known that if E is a locally convex space then $\text{Ba}(E_w) = \sigma(E')$, where E_w stands for E equipped with its weak topology and E' is the (topological) dual of E , see [9, Theorem 2.3]. In particular, we have:

- (i) $\text{Ba}(C_p(K)) = \sigma(\{\delta_t : t \in K\})$ for every compact space K , where δ_t denotes the Dirac delta at $t \in K$.
- (ii) $\text{Ba}(X_w) = \sigma(X^*)$ for every Banach space X (with dual X^*).

In view of (ii) and the Hahn-Banach theorem, if Y is a closed subspace of a Banach space X , then the trace of $\text{Ba}(X_w)$ on Y is exactly $\text{Ba}(Y_w)$.

2. THE MAIN RESULT

The aim of this section is to prove that the equality $\text{Ba}(C_p(2^\Gamma)) = \text{Bo}(C(2^\Gamma))$ is equivalent to saying that $|\Gamma|$ is a Kunen cardinal (Theorem 2.8 below). The proof is split into several lemmas for the convenience of the reader. Throughout this section Γ is a fixed infinite set.

Lemma 2.1. *Let $A \in \mathcal{P}(\Gamma) \otimes \mathcal{P}(\Gamma)$. Define an equivalence relation \approx on Γ by saying that $\gamma \approx \gamma'$ if and only if, for each $\delta \in \Gamma$, we have*

$$(\delta, \gamma) \in A \Leftrightarrow (\delta, \gamma') \in A \quad \text{and} \quad (\gamma, \delta) \in A \Leftrightarrow (\gamma', \delta) \in A.$$

Then \approx has at most \mathfrak{c} many equivalence classes.

Proof. Take $B_n \subset \Gamma$, $n \in \mathbb{N}$, such that A belongs to the σ -algebra \mathcal{A}_0 on Γ^2 generated by the sequence $(B_{2m} \times B_{2m-1})_{m \in \mathbb{N}}$. Define an equivalence relation \sim on Γ by

$$\gamma \sim \gamma' \Leftrightarrow 1_{B_n}(\gamma) = 1_{B_n}(\gamma') \text{ for all } n \in \mathbb{N}.$$

Since there are at most \mathfrak{c} distinct sequences of the form $(1_{B_n}(\gamma))_{n \in \mathbb{N}} \in 2^\mathbb{N}$, the relation \sim has at most \mathfrak{c} many equivalence classes. Let \mathcal{A}_1 be the family made up of all $C \in \mathcal{A}_0$ such that, for each $\gamma \sim \gamma'$ and $\delta \sim \delta'$, we have

$$(\gamma, \delta) \in C \Leftrightarrow (\gamma', \delta') \in C.$$

Clearly \mathcal{A}_1 is a σ -algebra containing $B_{2m} \times B_{2m-1}$ for all $m \in \mathbb{N}$, hence $\mathcal{A}_0 = \mathcal{A}_1$ and so $A \in \mathcal{A}_1$. In particular, we have $\gamma \approx \gamma'$ whenever $\gamma \sim \gamma'$. It follows that the relation \approx has at most \mathfrak{c} many equivalence classes as well. \square

Part (ii) of the following lemma is well-known, see [17].

Lemma 2.2. *Let $\Omega = \{(\gamma_1, \gamma_2) \in \Gamma^2 : \gamma_1 \neq \gamma_2\}$ and let Σ be the trace of $\mathcal{P}(\Gamma) \otimes \mathcal{P}(\Gamma)$ on Ω . Then:*

- (i) $|\Gamma|$ is a Kunen cardinal if and only if $\Sigma = \mathcal{P}(\Omega)$.
- (ii) If $|\Gamma| > \mathfrak{c}$, then $|\Gamma|$ is not a Kunen cardinal.

Proof. We distinguish two cases:

CASE $|\Gamma| \leq \mathfrak{c}$. We can assume without loss of generality that $\Gamma \subset \mathbb{R}$. For each $U \subset \Gamma$, we have

$$(2.1) \quad \{(\gamma, \gamma) : \gamma \in U\} = \bigcap_{n \in \mathbb{N}} \bigcup_{q \in \mathbb{Q}} \left(U \cap \left(q - \frac{1}{n}, q + \frac{1}{n} \right) \right)^2 \in \mathcal{P}(\Gamma) \otimes \mathcal{P}(\Gamma).$$

In particular, we get $\Omega \in \mathcal{P}(\Gamma) \otimes \mathcal{P}(\Gamma)$ and so $\Sigma \subset \mathcal{P}(\Gamma) \otimes \mathcal{P}(\Gamma)$. Suppose now that $|\Gamma|$ is not a Kunen cardinal. If $A \subset \Gamma^2$ is any set not belonging to $\mathcal{P}(\Gamma) \otimes \mathcal{P}(\Gamma)$, then $A \cap \Omega \notin \Sigma$ because (2.1) implies that $A \setminus \Omega \in \mathcal{P}(\Gamma) \otimes \mathcal{P}(\Gamma)$.

CASE $|\Gamma| > \mathfrak{c}$. Let \equiv be an equivalence relation on Γ for which all equivalence classes are infinite and have cardinality less than or equal to \mathfrak{c} . We shall check that the set

$$W := \{(\gamma_1, \gamma_2) \in \Omega : \gamma_1 \equiv \gamma_2\}$$

does not belong to Σ . Suppose if possible otherwise. Then there is $A \in \mathcal{P}(\Gamma) \otimes \mathcal{P}(\Gamma)$ such that $A \cap \Omega = W$. Let \approx be the equivalence relation on Γ induced by A as defined in Lemma 2.1. Since $|\Gamma| > \mathfrak{c}$, an appeal to Lemma 2.1 ensures the existence of $E \subset \Gamma$ with $|E| > \mathfrak{c}$ such that $\gamma \approx \gamma'$ whenever $\gamma, \gamma' \in E$. Given distinct $\gamma, \gamma' \in E$ we can find $\delta \in \Gamma \setminus \{\gamma, \gamma'\}$ with $\delta \equiv \gamma$. Then $(\delta, \gamma) \in W = A \cap \Omega$ and the fact that $\gamma \approx \gamma'$ implies that $(\delta, \gamma') \in A \cap \Omega = W$, hence $\gamma \equiv \gamma'$. This means that E is contained in some equivalence class of \equiv , which has cardinality less than or equal to \mathfrak{c} . This contradiction finishes the proof. \square

From now on we denote by \mathfrak{I} the family of all closed nonempty intervals of \mathbb{R} .

Definition 2.3. *Let $n \in \mathbb{N}$.*

- (i) *A function $\tau : 2^n \rightarrow \mathfrak{I}$ is called a type (or an n -type).*
- (ii) *Let τ be an n -type. We say that $f \in C(2^\Gamma)$ has type τ if there exist $\gamma_1, \dots, \gamma_n \in \Gamma$ such that*

$$f(x) \in \tau(x_{\gamma_1}, \dots, x_{\gamma_n}) \quad \text{for every } x \in 2^\Gamma.$$

We denote by Y_τ the set of all $f \in C(2^\Gamma)$ having type τ .

Lemma 2.4. *If $|\Gamma| \leq \mathfrak{c}$, then Y_τ belongs to $\text{Ba}(C_p(2^\Gamma))$ for every type τ .*

Proof. Since $|\Gamma| \leq \mathfrak{c}$, we can suppose that Γ is a subset of the Cantor set $\Delta = 2^\mathbb{N}$. We write $\gamma = (\gamma[m])_{m \in \mathbb{N}}$ when we express $\gamma \in \Delta$ as a sequence of 0's and 1's. For each $m \in \mathbb{N}$, we consider

$$\Gamma_m := \{\gamma \in \Delta : \gamma[k] = 0 \text{ for all } k > m\}.$$

Observe that $\bigcup_{m \in \mathbb{N}} \Gamma_m$ is countable and so we can suppose without loss of generality that $\bigcup_{m \in \mathbb{N}} \Gamma_m \subset \Gamma$. For each $m \in \mathbb{N}$, let

$$K_m := \{x \in 2^\Gamma : x_\gamma = x_\delta \text{ whenever } \gamma, \delta \in \Gamma \text{ satisfy } \gamma[k] = \delta[k] \text{ for all } k \leq m\}.$$

Note that K_m is finite. Indeed, it is easy to check that $K_m = \{x^\sigma : \sigma \in 2^{2^m}\}$, where $x^\sigma \in 2^\Gamma$ is defined by $x^\sigma(\gamma) := \sigma((\gamma[1], \dots, \gamma[m]))$ for all $\gamma \in \Gamma$.

Let $n \in \mathbb{N}$ be such that τ is an n -type. The set

$$A := \bigcap_{m \in \mathbb{N}} \bigcup_{\gamma_1^m, \dots, \gamma_n^m \in \Gamma_m} \bigcap_{x \in K_m} \{f \in C(2^\Gamma) : f(x) \in \tau(x_{\gamma_1^m}, \dots, x_{\gamma_n^m})\}$$

belongs to $\text{Ba}(C_p(2^\Gamma))$. So, in order to prove that $Y_\tau \in \text{Ba}(C_p(2^\Gamma))$ it is enough to check that $Y_\tau = A$.

We first prove $Y_\tau \subset A$. Take $f \in Y_\tau$. Then there exist $\gamma_1, \dots, \gamma_n \in \Gamma$ such that $f(x) \in \tau(x_{\gamma_1}, \dots, x_{\gamma_n})$ for every $x \in 2^\Gamma$. Given $m \in \mathbb{N}$ and $i \in \{1, \dots, n\}$, we can choose $\gamma_i^m \in \Gamma_m$ such that $\gamma_i^m[k] = \gamma_i[k]$ for all $k \leq m$. For each $x \in K_m$ we have $x_{\gamma_i^m} = x_{\gamma_i}$ and hence $f(x) \in \tau(x_{\gamma_1^m}, \dots, x_{\gamma_n^m})$. Therefore, $f \in A$.

We now prove $A \subset Y_\tau$. Take $f \in A$. We can consider the function $\tilde{f} \in C(2^\Delta)$ given by $\tilde{f}(x) := f(x|_\Gamma)$. For each $m \in \mathbb{N}$, set

$$\tilde{K}_m := \{x \in 2^\Delta : x_\gamma = x_\delta \text{ whenever } \gamma, \delta \in \Delta \text{ satisfy } \gamma[k] = \delta[k] \text{ for all } k \leq m\},$$

$$P_m = \{(\gamma_1, \dots, \gamma_n) \in \Delta^n : \tilde{f}(x) \in \tau(x_{\gamma_1}, \dots, x_{\gamma_n}) \text{ for all } x \in \tilde{K}_m\}.$$

Observe that $P_m \neq \emptyset$ because $f \in A$ and $x|_\Gamma \in K_m$ whenever $x \in \tilde{K}_m$. It is easy to check that, for each $x \in \tilde{K}_m$, the set $\{(\gamma_1, \dots, \gamma_n) \in \Delta^n : \tilde{f}(x) \in \tau(x_{\gamma_1}, \dots, x_{\gamma_n})\}$ is closed, hence P_m is compact. Now, since $P_m \supset P_{m+1}$ for all $m \in \mathbb{N}$, we can pick $(\delta_1, \dots, \delta_n) \in \bigcap_{m \in \mathbb{N}} P_m$. Then $\tilde{f}(x) \in \tau(x_{\delta_1}, \dots, x_{\delta_n})$ for every $x \in \bigcup_{m \in \mathbb{N}} \tilde{K}_m$.

We claim that $\bigcup_{m \in \mathbb{N}} \tilde{K}_m$ is dense in 2^Δ . Indeed, fix $z \in 2^\Delta$ and take a finite set of coordinates $\{\gamma_1, \dots, \gamma_p\} \subset \Delta$. Choose $m \in \mathbb{N}$ large enough such that $(\gamma_i[1], \dots, \gamma_i[m]) \neq (\gamma_j[1], \dots, \gamma_j[m])$ whenever $i \neq j$. Then the element $x \in 2^\Delta$ defined by

$$x_\gamma := \begin{cases} z_{\gamma_i} & \text{if } \gamma[k] = \gamma_i[k] \text{ for all } k \leq m, \\ 0 & \text{otherwise,} \end{cases}$$

belongs to \tilde{K}_m and satisfies $x_{\gamma_i} = z_{\gamma_i}$ for every i . This proves the claim.

It follows that $\tilde{f}(x) \in \tau(x_{\delta_1}, \dots, x_{\delta_n})$ for every $x \in 2^\Delta$. We choose an arbitrary $\xi \in \Gamma$ and, for each $i \in \{1, \dots, n\}$, we define $\gamma_i := \delta_i$ if $\delta_i \in \Gamma$ and $\gamma_i := \xi$ if $\delta_i \notin \Gamma$. We claim that $f(x) \in \tau(x_{\gamma_1}, \dots, x_{\gamma_n})$ for every $x \in 2^\Gamma$. Indeed, given any $x \in 2^\Gamma$, we can select $z \in 2^\Delta$ such that $z|_\Gamma = x$ and $z_{\delta_i} = x_\xi$ whenever $\delta_i \notin \Gamma$, so that

$$f(x) = \tilde{f}(z) \in \tau(z_{\delta_1}, \dots, z_{\delta_n}) = \tau(x_{\gamma_1}, \dots, x_{\gamma_n}),$$

as claimed. This shows that $f \in Y_\tau$ and the proof is over. \square

The proof of the key Lemma 2.7 is rather technical and will be given later (Subsection 2.1). In order to state that lemma we first need some definitions. From now on, the “coordinates” of any $\gamma \in \Gamma^n$, $n \in \mathbb{N}$, are denoted by $\gamma_1, \dots, \gamma_n$, that is, we write $\gamma = (\gamma_1, \dots, \gamma_n)$.

Definition 2.5. *Let τ be an n -type.*

- (i) *We say that $\gamma, \delta \in \Gamma^n$ are τ -proximal if*

$$\tau(1_U(\gamma_1), \dots, 1_U(\gamma_n)) \cap \tau(1_U(\delta_1), \dots, 1_U(\delta_n)) \neq \emptyset$$

for every $U \subset \Gamma$.

- (ii) *We say that $A, B \subset \Gamma^n$ are τ -separated if there exist no $\gamma \in A$ and $\delta \in B$ which are τ -proximal.*

Definition 2.6. *Let (Y, Σ) be a measurable space. We say that $U, V \subset Y$ are Σ -separated if there is $S \in \Sigma$ such that $U \subset S$ and $V \cap S = \emptyset$.*

Lemma 2.7. *Let τ be an n -type, (Y, Σ) a measurable space and $\Phi : \Gamma^n \rightarrow \mathcal{P}(Y)$ a multifunction satisfying:*

- (S) *For each $U \subset \Gamma$ and each closed set $I \subset \mathbb{R}$, the sets*

$$\Phi(\{\gamma \in \Gamma^n : \tau(1_U(\gamma_1), \dots, 1_U(\gamma_n)) \subset I\})$$

$$\Phi(\{\gamma \in \Gamma^n : \tau(1_U(\gamma_1), \dots, 1_U(\gamma_n)) \cap I = \emptyset\})$$

are Σ -separated.

Suppose $|\Gamma|$ is a Kunen cardinal. If $A, B \subset \Gamma^n$ are τ -separated, then $\Phi(A)$ and $\Phi(B)$ are Σ -separated.

We write $C(2^\Gamma, 2)$ to denote the subset of $C(2^\Gamma)$ made up of all $\{0, 1\}$ -valued functions, which can be identified with the algebra $\text{Clop}(2^\Gamma)$ of all clopen subsets of 2^Γ via the bijection

$$\psi : \text{Clop}(2^\Gamma) \rightarrow C(2^\Gamma, 2), \quad \psi(A) := 1_A.$$

The trace of $\text{Ba}(C_p(2^\Gamma))$ on $C(2^\Gamma, 2)$ is denoted by $\text{Ba}(C_p(2^\Gamma, 2))$. Observe that $\{\psi^{-1}(E) : E \in \text{Ba}(C_p(2^\Gamma, 2))\}$ is exactly the σ -algebra on $\text{Clop}(2^\Gamma)$ generated by all ultrafilters. On the other hand, since $C(2^\Gamma, 2)$ is norm discrete, the trace of $\text{Bo}(C(2^\Gamma))$ on $C(2^\Gamma, 2)$ is exactly $\mathcal{P}(C(2^\Gamma, 2))$.

We now arrive at our main result:

Theorem 2.8. *The following statements are equivalent:*

- (i) $|\Gamma|$ is a Kunen cardinal.
- (ii) $\text{Ba}(C_p(2^\Gamma)) = \text{Bo}(C(2^\Gamma))$.
- (iii) $\text{Ba}(C_p(2^\Gamma, 2)) = \mathcal{P}(C(2^\Gamma, 2))$.
- (iv) *The σ -algebra on $\text{Clop}(2^\Gamma)$ generated by all ultrafilters is $\mathcal{P}(\text{Clop}(2^\Gamma))$.*

Proof. (iii) \Leftrightarrow (iv) follows from the comments preceding the theorem.

(i) \Rightarrow (ii). Let us write $Y := C(2^\Gamma)$ and $\Sigma := \text{Ba}(C_p(2^\Gamma))$. Let Θ be an open subset of Y in the norm topology. We shall prove that $\Theta \in \Sigma$.

STEP 1. Fix an n -type τ and consider the multifunction $\Phi^\tau : \Gamma^n \rightarrow \mathcal{P}(Y)$ given by

$$\Phi^\tau(\gamma) := \{f \in Y : f(x) \in \tau(x_{\gamma_1}, \dots, x_{\gamma_n}) \text{ for all } x \in 2^\Gamma\} \subset Y_\tau.$$

We first observe that $\gamma, \delta \in \Gamma^n$ are τ -proximal if and only if $\Phi^\tau(\gamma) \cap \Phi^\tau(\delta) \neq \emptyset$. Indeed, the “if” part follows from the fact that

$$f(1_U) \in \tau(1_U(\gamma_1), \dots, 1_U(\gamma_n)) \cap \tau(1_U(\delta_1), \dots, 1_U(\delta_n))$$

whenever $f \in \Phi^\tau(\gamma) \cap \Phi^\tau(\delta)$ and $U \subset \Gamma$. Conversely, assume that γ and δ are τ -proximal. Then for each $U \subset \Gamma$ we can pick

$$t_U \in \tau(1_U(\gamma_1), \dots, 1_U(\gamma_n)) \cap \tau(1_U(\delta_1), \dots, 1_U(\delta_n)).$$

Let W be the subset of Γ made up of all γ_i 's and δ_i 's. Since W is finite, the function $f : 2^\Gamma \rightarrow \mathbb{R}$ given by $f(1_U) := t_{U \cap W}$ is continuous. Moreover, since $1_U(\gamma_i) = 1_{U \cap W}(\gamma_i)$ and $1_U(\delta_i) = 1_{U \cap W}(\delta_i)$ for every $U \subset \Gamma$ and every i , we have $f \in \Phi^\tau(\gamma) \cap \Phi^\tau(\delta)$. Hence $\Phi^\tau(\gamma) \cap \Phi^\tau(\delta) \neq \emptyset$.

It follows at once that the following two subsets of Γ^n are τ -separated:

$$A_\tau := \{\gamma \in \Gamma^n : \Phi^\tau(\gamma) \setminus \Theta \neq \emptyset\},$$

$$B_\tau = \{\gamma \in \Gamma^n : \Phi^\tau(\gamma) \cap \Phi^\tau(A_\tau) = \emptyset\}.$$

On the other hand, $Y_\tau \in \Sigma$ (by Lemmas 2.2 and 2.4) and so, for each $U \subset \Gamma$ and each closed set $I \subset \mathbb{R}$, the set $S_{(U,I)} := \{f \in Y_\tau : f(1_U) \in I\}$ belongs to Σ and satisfies

$$\Phi^\tau(\{\gamma \in \Gamma^n : \tau(1_U(\gamma_1), \dots, 1_U(\gamma_n)) \subset I\}) \subset S_{(U,I)},$$

$$\Phi^\tau(\{\gamma \in \Gamma^n : \tau(1_U(\gamma_1), \dots, 1_U(\gamma_n)) \cap I = \emptyset\}) \cap S_{(U,I)} = \emptyset.$$

An appeal to Lemma 2.7 ensures that $\Phi^\tau(A_\tau)$ and $\Phi^\tau(B_\tau)$ are Σ -separated, that is, there is $\Theta_\tau \in \Sigma$ such that $\Phi^\tau(B_\tau) \subset \Theta_\tau$ and $\Phi^\tau(A_\tau) \cap \Theta_\tau = \emptyset$. Bearing in mind that $Y_\tau \in \Sigma$, we can assume further that $\Theta_\tau \subset Y_\tau$.

STEP 2. We write \mathfrak{I}_0 to denote the (countable) family of all closed nonempty intervals of \mathbb{R} with rational endpoints. To finish the proof we shall check that

$$(2.2) \quad \Theta = \bigcup \{\Theta_\tau : \tau \text{ is a type with values in } \mathfrak{I}_0\}.$$

On the one hand, for any n -type τ , we have $\Theta_\tau \subset Y_\tau \setminus \Phi^\tau(A_\tau)$. Moreover, we have $Y_\tau \setminus \Phi^\tau(A_\tau) \subset \Theta$, because for each $f \in Y_\tau \setminus \Phi^\tau(A_\tau)$ there is some $\gamma \in \Gamma^n \setminus A_\tau$ such that $f \in \Phi^\tau(\gamma) \subset \Theta$. Thus, the inclusion “ \supset ” in (2.2) holds true.

In order to prove the reverse inclusion, fix $f \in \Theta$. Since Θ is norm open, there is $\varepsilon > 0$ such that $\|f - h\|_\infty \geq 2\varepsilon$ for every $h \in Y \setminus \Theta$. By the continuity of f and the compactness of 2^Γ , we can find finitely many basic clopen sets $C_i \subset 2^\Gamma$ such that $2^\Gamma = \bigcup_i C_i$ and the oscillation of f on each C_i is less than ε . Thus, we can find a finite set $\{\gamma_1, \dots, \gamma_n\} \subset \Gamma$ and a type $\tau : 2^n \rightarrow \mathfrak{I}_0$ such that:

- (a) $\tau(p)$ has length less than ε for every $p \in 2^n$,
- (b) $f(x) \in \tau(x_{\gamma_1}, \dots, x_{\gamma_n})$ for every $x \in 2^\Gamma$.

Condition (b) means that $f \in \Phi^\tau(\gamma)$, where $\gamma := (\gamma_1, \dots, \gamma_n) \in \Gamma^n$.

We claim that $f \in \Theta_\tau$. Indeed, it suffices to check that $\gamma \in B_\tau$, because in that case we would have $f \in \Phi^\tau(\gamma) \subset \Phi^\tau(B_\tau) \subset \Theta_\tau$. Our proof is by contradiction: suppose that $\gamma \notin B_\tau$. Then there exists $\delta \in A_\tau$ such that $\Phi^\tau(\gamma) \cap \Phi^\tau(\delta) \neq \emptyset$. Take $g \in \Phi^\tau(\gamma) \cap \Phi^\tau(\delta)$ and $h \in \Phi^\tau(\delta) \setminus \Theta$. By (a) we have:

$$\|u - v\|_\infty < \varepsilon \quad \text{for every } u, v \in \Phi^\tau(\zeta) \text{ and every } \zeta \in \Gamma^n.$$

Therefore, $\|f - g\|_\infty < \varepsilon$ (since $f, g \in \Phi^\tau(\gamma)$) and $\|g - h\|_\infty < \varepsilon$ (since $g, h \in \Phi^\tau(\delta)$). We conclude that $\|f - h\|_\infty < 2\varepsilon$, which contradicts the choice of ε because $h \notin \Theta$.

(ii) \Rightarrow (iii) is obvious.

(iii) \Rightarrow (i). Let $\Omega := \{(\gamma_1, \gamma_2) \in \Gamma^2 : \gamma_1 \neq \gamma_2\}$ be equipped with the trace Σ of the product σ -algebra $\mathcal{P}(\Gamma) \otimes \mathcal{P}(\Gamma)$. The function $H : \Omega \rightarrow C(2^\Gamma, 2)$ given by

$$H(\gamma_1, \gamma_2)(x) := x_{\gamma_1}(1 - x_{\gamma_2})$$

is Σ -Ba($C_p(2^\Gamma, 2)$)-measurable, because for each $x \in 2^\Gamma$ we have

$$\{(\gamma_1, \gamma_2) \in \Omega : H(\gamma_1, \gamma_2)(x) = 1\} = \{\gamma \in \Gamma : x_\gamma = 1\} \times \{\gamma \in \Gamma : x_\gamma = 0\} \in \Sigma.$$

Since $\text{Ba}(C_p(2^\Gamma, 2)) = \mathcal{P}(C(2^\Gamma, 2))$, we have $H^{-1}(X) \in \Sigma$ for every $X \subset C_p(2^\Gamma, 2)$. Thus, bearing in mind that H is one-to-one, we conclude that $\Sigma = \mathcal{P}(\Omega)$. An appeal to Lemma 2.2(i) ensures that $|\Gamma|$ is a Kunen cardinal. The proof is over. \square

Recall that a compact space K is called *dyadic* if K is a continuous image of 2^κ for some cardinal κ ; in this case, κ can be taken to be equal to the weight of K , see [11, 3.12.12]. The class of dyadic compacta of (infinite) weight κ contains in particular κ -fold products of compact metrizable spaces.

Corollary 2.9. *If K is a dyadic space and its weight is a Kunen cardinal, then $\text{Ba}(C_p(K)) = \text{Bo}(C(K))$.*

Proof. Let κ be the weight of K . If $\varphi : 2^\kappa \rightarrow K$ is a continuous surjection then the mapping $T : C(K) \rightarrow C(2^\kappa)$, $T(g) := g \circ \varphi$, is an isometric embedding which is pointwise continuous, so the assertion follows directly from Theorem 2.8. \square

Corollary 2.10. *Let $\{X_\alpha : \alpha < \kappa\}$ be a family of separable Banach spaces, where κ is a Kunen cardinal. Then $X := \bigoplus_{\ell^1} \{X_\alpha : \alpha < \kappa\}$ satisfies $\text{Ba}(X_w) = \text{Bo}(X)$.*

Proof. If κ is finite then X is separable and so $\text{Ba}(X_w) = \text{Bo}(X)$. Suppose κ is infinite. Since each $(B_{X_\alpha^*}, w^*)$ is a metrizable compact, there is a continuous surjection $2^\mathbb{N} \rightarrow B_{X_\alpha^*}$. Hence there is a continuous surjection

$$2^\kappa \rightarrow \prod_{\alpha < \kappa} B_{X_\alpha^*} = B_{X^*},$$

so X is isometric to a closed subspace of $C(2^\kappa)$. Since $\text{Ba}(C(2^\kappa)_w) = \text{Bo}(C(2^\kappa))$ (by Theorem 2.8), we have $\text{Ba}(X_w) = \text{Bo}(X)$ as well. \square

Corollary 2.11 (Fremlin). *$\text{Ba}(\ell^1(\Gamma)_w) = \text{Bo}(\ell^1(\Gamma))$ if $|\Gamma|$ is a Kunen cardinal.*

Remark 2.12. Let K be a compact space.

- (i) Suppose there exists a maximal family $\{\mu_\alpha : \alpha < \kappa\}$ of mutually singular Radon probabilities on K such that:

- κ is a Kunen cardinal,
- each $L^1(\mu_\alpha)$ is separable.

Then $\text{Ba}(C(K)_w^*) = \text{Bo}(C(K)^*)$, because $C(K)^*$ is isomorphic to the space $\bigoplus_{\ell^1} \{L^1(\mu_\alpha) : \alpha < \kappa\}$ (cf. [1, proof of Proposition 4.3.8]).

- (ii) The existence of a family $\{\mu_\alpha : \alpha < \kappa\}$ as in (i) is guaranteed if:

- $|K| = \mathfrak{c}$ is Kunen,
- $\text{span}\{\delta_t : t \in K\}$ is sequentially w^* -dense in $C(K)^*$,
- $L^1(\mu)$ is separable for every Radon probability μ on K .

Thus, assuming that \mathfrak{c} is Kunen, the equality $\text{Ba}(C(K)_w^*) = \text{Bo}(C(K)^*)$ holds true whenever $|K| = \mathfrak{c}$ and K belongs to one of the following classes of compacta: Eberlein, Corson (under MA + non CH), Rosenthal, linearly ordered, Radon-Nikodým, etc. (see e.g. [8, 21] and the references therein).

2.1. Proof of Lemma 2.7. This subsection is devoted to prove Lemma 2.7 above. The proof is divided into several auxiliary lemmas. Throughout, τ is an n -type, (Y, Σ) is a measurable space and $\Phi : \Gamma^n \rightarrow \mathcal{P}(Y)$ is a multifunction satisfying:

- (S) For each $U \subset \Gamma$ and each closed set $I \subset \mathbb{R}$, the sets

$$\begin{aligned} \Phi(\{\gamma \in \Gamma^n : \tau(1_U(\gamma_1), \dots, 1_U(\gamma_n)) \subset I\}) \\ \Phi(\{\gamma \in \Gamma^n : \tau(1_U(\gamma_1), \dots, 1_U(\gamma_n)) \cap I = \emptyset\}) \end{aligned}$$

are Σ -separated.

Definition 2.13. Let E be an equivalence relation on $\{1, \dots, n\} \times \{0, 1\}$. We say that E is a τ -proximity relation (and we write $E \in \text{Prox}(\tau)$) if $\tau(\gamma^0) \cap \tau(\gamma^1) \neq \emptyset$ whenever $\gamma^0, \gamma^1 \in 2^n$ satisfy

$$(p, i)E(q, j) \Rightarrow \gamma_p^i = \gamma_q^j$$

for every $(p, i), (q, j) \in \{1, \dots, n\} \times \{0, 1\}$.

Lemma 2.14. Let $\gamma^0, \gamma^1 \in \Gamma^n$. The following statements are equivalent:

- (i) γ^0, γ^1 are τ -proximal.
- (ii) There is $E \in \text{Prox}(\tau)$ such that

$$(p, i)E(q, j) \Rightarrow \gamma_p^i = \gamma_q^j$$

for every $(p, i), (q, j) \in \{1, \dots, n\} \times \{0, 1\}$.

Proof. (i) \Rightarrow (ii). The equivalence relation E on $\{1, \dots, n\} \times \{0, 1\}$ defined by

$$(p, i)E(q, j) \Leftrightarrow \gamma_p^i = \gamma_q^j$$

is a τ -proximity relation. Indeed, let $\delta^0, \delta^1 \in 2^n$ satisfy the condition:

$$(p, i)E(q, j) \Rightarrow \delta_p^i = \delta_q^j$$

for every $(p, i), (q, j) \in \{1, \dots, n\} \times \{0, 1\}$. Let $U \subset \Gamma$ be the set made up of all γ_p^0 's with $\delta_p^0 = 1$ and all γ_p^1 's with $\delta_p^1 = 1$. Then

$$\tau(1_U(\gamma_1^i), \dots, 1_U(\gamma_n^i)) = \tau(\delta^i) \quad \text{for } i \in \{0, 1\}$$

and so the τ -proximality of γ^0 and γ^1 implies that $\tau(\delta^0) \cap \tau(\delta^1) \neq \emptyset$.

(ii) \Rightarrow (i). Fix $U \subset \Gamma$ and set

$$\delta^i := (1_U(\gamma_1^i), \dots, 1_U(\gamma_n^i)) \in 2^n \quad \text{for } i \in \{0, 1\}.$$

Observe that if $(p, i)E(q, j)$ then $\gamma_p^i = \gamma_q^j$ and so $1_U(\gamma_p^i) = 1_U(\gamma_q^j)$. Bearing in mind that $E \in \text{Prox}(\tau)$, we conclude that

$$\tau(1_U(\gamma_1^0), \dots, 1_U(\gamma_n^0)) \cap \tau(1_U(\gamma_1^1), \dots, 1_U(\gamma_n^1)) = \tau(\delta^0) \cap \tau(\delta^1) \neq \emptyset.$$

This shows that γ^0 and γ^1 are τ -proximal. \square

Definition 2.15. Let $E \in \text{Prox}(\tau)$.

- (i) An equivalence class \mathcal{C} of E is called a linking class if $\mathcal{C} = [(p, 0)] = [(q, 1)]$ for some $p, q \in \{1, \dots, n\}$. We denote by ℓ_E the set of linking equivalence classes of E .
- (ii) Let $i \in \{0, 1\}$ and $A \subset \Gamma^n$. We define $L_E^i(A)$ as the set of all $\tilde{\gamma} \in \Gamma^{\ell_E}$ for which there is $\gamma \in A$ such that:
 - $\gamma_p = \gamma_q$ whenever $(p, i)E(q, i)$;
 - $\gamma_k = \tilde{\gamma}_{[(k, i)]}$ whenever $[(k, i)] \in \ell_E$.

Lemma 2.16. Let $A, B \subset \Gamma^n$. The following statements are equivalent:

- (i) A and B are τ -separated.
- (ii) $L_E^0(A) \cap L_E^1(B) = \emptyset$ for every $E \in \text{Prox}(\tau)$.

Proof. (i) \Rightarrow (ii). Suppose that $L_E^0(A) \cap L_E^1(B) \neq \emptyset$ for some $E \in \text{Prox}(\tau)$. Take $\tilde{\gamma} \in L_E^0(A) \cap L_E^1(B)$ and choose $\gamma^0 \in A$, $\gamma^1 \in B$, such that for $i \in \{0, 1\}$ we have

$$\gamma_p^i = \gamma_q^i \text{ whenever } (p, i)E(q, i) \text{ and } \gamma_k^i = \tilde{\gamma}_{[(k, i)]} \text{ for every } [(k, i)] \in \ell_E.$$

Therefore, $\gamma_p^i = \gamma_q^j$ whenever $(p, i)E(q, j)$. An appeal to Lemma 2.14 ensures that γ^0 and γ^1 are τ -proximal, so A and B are not τ -separated.

(ii) \Rightarrow (i). If A and B are not τ -separated, then (by Lemma 2.14) there exist $\gamma^0 \in A$, $\gamma^1 \in B$ and $E \in \text{Prox}(\tau)$ such that

$$(p, i)E(q, j) \Rightarrow \gamma_p^i = \gamma_q^j$$

for every $(p, i), (q, j) \in \{1, \dots, n\} \times \{0, 1\}$. Then we can define $\tilde{\gamma} \in \Gamma^{\ell_E}$ by saying that $\tilde{\gamma}_{[(p, i)]} := \gamma_p^i$ for every $[(p, i)] \in \ell_E$. Clearly, $\tilde{\gamma} \in L_E^0(A) \cap L_E^1(B)$. \square

Remark 2.17. Let $U_n, V_n \subset Y$, $n \in \mathbb{N}$. If U_n and V_m are Σ -separated for every $n, m \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} U_n$ and $\bigcup_{n \in \mathbb{N}} V_n$ are Σ -separated as well.

Proof. For each $n, m \in \mathbb{N}$, fix $S_{n, m} \in \Sigma$ such that $U_n \subset S_{n, m}$ and $V_m \cap S_{n, m} = \emptyset$. Then $S := \bigcup_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} S_{n, m} \in \Sigma$ satisfies $\bigcup_{n \in \mathbb{N}} U_n \subset S$ and $(\bigcup_{n \in \mathbb{N}} V_n) \cap S = \emptyset$. \square

Lemma 2.18. Let $E_0 \in \text{Prox}(\tau)$. For each $E \in \text{Prox}(\tau) \setminus \{E_0\}$, let us fix disjoint sets $X_E, Y_E \subset \Gamma^{\ell_E}$. Let \mathfrak{W} be the family of all $W \subset \Gamma^{\ell_{E_0}}$ for which the following statement holds:

- “If $A, B \subset \Gamma^n$ satisfy
 - $L_E^0(A) \subset X_E$ and $L_E^1(B) \subset Y_E$ for every $E \in \text{Prox}(\tau) \setminus \{E_0\}$,
 - $L_{E_0}^0(A) \subset W$ and $L_{E_0}^1(B) \cap W = \emptyset$,

then $\Phi(A)$ and $\Phi(B)$ are Σ -separated."

Then \mathfrak{V} is closed under countable unions and countable intersections.

Proof. Let $(W_m)_{m \in \mathbb{N}}$ be an arbitrary sequence in \mathfrak{V} . We shall prove first that $W := \bigcup_{m \in \mathbb{N}} W_m \in \mathfrak{V}$. For let $A, B \subset \Gamma^n$ be sets satisfying

- (i) $L_E^0(A) \subset X_E$ and $L_E^1(B) \subset Y_E$ for every $E \in \text{Prox}(\tau) \setminus \{E_0\}$,
- (ii) $L_{E_0}^0(A) \subset W$ and $L_{E_0}^1(B) \cap W = \emptyset$.

Note that for every $\gamma \in \Gamma^n$ the set $L_{E_0}^0(\{\gamma\})$ is either empty or a singleton. For each $m \in \mathbb{N}$, define

$$A_m := \{\gamma \in A : L_{E_0}^0(\{\gamma\}) \subset W_m\}.$$

Since $\bigcup_{\gamma \in A} L_{E_0}^0(\{\gamma\}) = L_{E_0}^0(A) \subset W$, we have $A = \bigcup_{m \in \mathbb{N}} A_m$. Thus, bearing in mind Remark 2.17, in order to prove that $\Phi(A) = \bigcup_{m \in \mathbb{N}} \Phi(A_m)$ and $\Phi(B)$ are Σ -separated it suffices to check that, for each $m \in \mathbb{N}$, the sets $\Phi(A_m)$ and $\Phi(B)$ are Σ -separated. Fix $m \in \mathbb{N}$ and observe that:

- $L_E^0(A_m) \subset L_E^0(A) \subset X_E$ and $L_E^1(B) \subset Y_E$ for $E \in \text{Prox}(\tau) \setminus \{E_0\}$ (by (i)),
- $L_{E_0}^0(A_m) = \bigcup_{\gamma \in A_m} L_{E_0}^0(\{\gamma\}) \subset W_m$ and $L_{E_0}^1(B) \cap W_m = \emptyset$ (by (ii)).

Since $W_m \in \mathfrak{V}$ we conclude that $\Phi(A_m)$ and $\Phi(B)$ are Σ -separated, as desired. It follows that $W \in \mathfrak{V}$.

We now prove that $W' := \bigcap_{m \in \mathbb{N}} W_m \in \mathfrak{V}$. Fix $A, B \subset \Gamma^n$ such that

- (i') $L_E^0(A) \subset X_E$ and $L_E^1(B) \subset Y_E$ for every $E \in \text{Prox}(\tau) \setminus \{E_0\}$,
- (ii') $L_{E_0}^0(A) \subset W'$ and $L_{E_0}^1(B) \cap W' = \emptyset$.

For each $m \in \mathbb{N}$ we define

$$B_m := \{\gamma \in B : L_{E_0}^1(\{\gamma\}) \cap W_m = \emptyset\}.$$

Since each $L_{E_0}^1(\{\gamma\})$ is either empty or a singleton, and

$$\bigcup_{\gamma \in B} L_{E_0}^1(\{\gamma\}) = L_{E_0}^1(B) \subset \Gamma^{\ell_{E_0}} \setminus W' = \bigcup_{m \in \mathbb{N}} \Gamma^{\ell_{E_0}} \setminus W_m,$$

we have $B = \bigcup_{m \in \mathbb{N}} B_m$. Therefore, to show that $\Phi(A)$ and $\Phi(B) = \bigcup_{m \in \mathbb{N}} \Phi(B_m)$ are Σ -separated it is enough to check that, for each $m \in \mathbb{N}$, the sets $\Phi(A)$ and $\Phi(B_m)$ are Σ -separated. This follows immediately from the facts that $W_m \in \mathfrak{V}$ and

- $L_E^0(A) \subset X_E$ and $L_E^1(B_m) \subset L_E^1(B) \subset Y_E$ for $E \in \text{Prox}(\tau) \setminus \{E_0\}$ (by (i')),
- $L_{E_0}^0(A) \subset W' \subset W_m$ (by (ii')) and

$$L_{E_0}^1(B_m) = \bigcup_{\gamma \in B_m} L_{E_0}^1(\{\gamma\}) \subset \Gamma^{\ell_{E_0}} \setminus W_m.$$

This proves that $W' \in \mathfrak{V}$ and we are done. \square

Definition 2.19. Let Ω be a set and $A_1, \dots, A_m \in \mathcal{P}(\Omega)$. We say that $C \subset \Omega$ is an atom of the algebra on Ω generated by A_1, \dots, A_m if C is nonempty and can be written as $C = \bigcap_{i=1}^m D_i$ where each $D_i \in \{A_i, \Omega \setminus A_i\}$.

Definition 2.20. A set $W \subset \Gamma^n$ is called a product if it can be expressed as $W = \prod_{i=1}^n W_i$ for some $W_i \subset \Gamma$ (which are called the factors of W).

Lemma 2.21. *Let $A, B \subset \Gamma^n$ be products. If A and B are τ -separated, then $\Phi(A)$ and $\Phi(B)$ are Σ -separated.*

Proof. Write $A = \prod_{i=1}^n W_i$ and $B = \prod_{i=1}^n W'_i$. Let V_1, \dots, V_m be the atoms of the algebra on Γ generated by W_1, \dots, W_n and W'_1, \dots, W'_n . Then A (resp. B) is the union of all products of the form $\prod_{i=1}^n V_{k_i}$ where $V_{k_i} \subset W_i$ (resp. $V_{k_i} \subset W'_i$). Thus, an appeal to Remark 2.17 allows us to assume that A and B are of the form

$$A = \prod_{i=1}^n V_{k_i} \quad B = \prod_{i=1}^n V_{r_i}$$

for some $k_i, r_i \in \{1, \dots, m\}$.

For each $j = 1, \dots, m$ we choose $\gamma_j \in V_j$. Define $\gamma^0 \in A$ and $\gamma^1 \in B$ by declaring $\gamma_i^0 := \gamma_{k_i}$ and $\gamma_i^1 := \gamma_{r_i}$ for $i \in \{1, \dots, n\}$. Since A and B are τ -separated, γ^0 and γ^1 are not τ -proximal, so there exists $U \subset \Gamma$ such that

$$\tau(1_U(\gamma_{k_1}), \dots, 1_U(\gamma_{k_n})) \cap \tau(1_U(\gamma_{r_1}), \dots, 1_U(\gamma_{r_n})) = \emptyset.$$

Define $V := \bigcup \{V_j : \gamma_j \in U\} \subset \Gamma$. Observe that for each $i \in \{1, \dots, n\}$ we have $\gamma_{k_i} \in U$ if and only if $\gamma_{k_i} \in V$, and $\gamma_{r_i} \in U$ if and only if $\gamma_{r_i} \in V$. Therefore

$$(2.3) \quad \tau(1_V(\gamma_{k_1}), \dots, 1_V(\gamma_{k_n})) \cap \tau(1_V(\gamma_{r_1}), \dots, 1_V(\gamma_{r_n})) = \emptyset.$$

Set $I := \tau(1_V(\gamma_{k_1}), \dots, 1_V(\gamma_{k_n})) \subset \mathbb{R}$. Observe that for each $\delta \in A = \prod_{i=1}^n V_{k_i}$ and each $i \in \{1, \dots, n\}$, we have $\delta_i \in V$ if and only if $V_{k_i} \subset V$, which is equivalent to saying that $\gamma_{k_i} \in V$. In particular,

$$A \subset \{\delta \in \Gamma^n : \tau(1_V(\delta_1), \dots, 1_V(\delta_n)) \subset I\}.$$

In the same way, bearing in mind (2.3) we have

$$B \subset \{\delta \in \Gamma^n : \tau(1_V(\delta_1), \dots, 1_V(\delta_n)) \cap I = \emptyset\}.$$

Now, property (S) of Φ implies that $\Phi(A)$ and $\Phi(B)$ are Σ -separated. \square

Throughout the rest of the subsection we assume that $|\Gamma| \leq \mathfrak{c}$, which is weaker than being a Kunen cardinal (Lemma 2.2(ii)). We can suppose without loss of generality that $\Gamma \subset \mathbb{R}$, so that Γ^n is equipped with the topology inherited from \mathbb{R}^n .

Lemma 2.22. *Let $A, B \subset \Gamma^n$ be open sets. If A and B are τ -separated, then $\Phi(A)$ and $\Phi(B)$ are Σ -separated.*

Proof. Let $\mathcal{O}_A, \mathcal{O}_B \subset \mathbb{R}^n$ be open sets such that $A = \Gamma^n \cap \mathcal{O}_A$ and $B = \Gamma^n \cap \mathcal{O}_B$. Both $\mathcal{O}_A, \mathcal{O}_B$ are countable unions of (open) products in \mathbb{R}^n and, therefore, we can write $A = \bigcup_{m \in \mathbb{N}} A_m$ and $B = \bigcup_{m \in \mathbb{N}} B_m$, where A_m and B_m are products in Γ^n . For each $k, m \in \mathbb{N}$ the sets A_k and B_m are τ -separated and Lemma 2.21 ensures that $\Phi(A_k)$ and $\Phi(B_m)$ are Σ -separated. Hence the sets $\Phi(A) = \bigcup_{m \in \mathbb{N}} \Phi(A_m)$ and $\Phi(B) = \bigcup_{m \in \mathbb{N}} \Phi(B_m)$ are Σ -separated (by Remark 2.17), as required. \square

Remark 2.23. The algebra on Γ^n generated by products is exactly the collection of all subsets of Γ^n which can be written as a disjoint union of finitely many products.

Proof. Let us write \mathcal{A} to denote such collection. In order to prove that \mathcal{A} is an algebra, observe first that \mathcal{A} is closed under finite intersections. On the other hand, given any product $W = \prod_{i=1}^n W_i$, then $\Gamma^n \setminus W$ is the disjoint union of all products of the form $\prod_{i=1}^n C_i$, where each C_i is an atom of the algebra on Γ generated by W_1, \dots, W_n and at least one C_i is disjoint from W_i . So, $\Gamma^n \setminus W \in \mathcal{A}$. It follows that \mathcal{A} is also closed under complements. \square

Lemma 2.24. *Let $A, B \subset \Gamma^n$ be such that for each $E \in \text{Prox}(\tau)$ there is $W_E \subset \Gamma^{\ell_E}$ in the algebra generated by products such that $L_E^0(A) \subset W_E$ and $L_E^1(B) \cap W_E = \emptyset$. Then $\Phi(A)$ and $\Phi(B)$ are Σ -separated.*

Proof. We divide the proof into several steps.

STEP 1. For each $E \in \text{Prox}(\tau)$, the set W_E (resp. $\Gamma^{\ell_E} \setminus W_E$) is the union of a finite collection \mathfrak{P}_E (resp. \mathfrak{Q}_E) of products in Γ^{ℓ_E} (Remark 2.23). Observe also that $\text{Prox}(\tau)$ is finite. Let C_1, \dots, C_m be the atoms of the algebra on Γ generated by the factors of all elements of the collection $\bigcup \{\mathfrak{P}_E \cup \mathfrak{Q}_E : E \in \text{Prox}(\tau)\}$. Then each W_E (resp. $\Gamma^{\ell_E} \setminus W_E$) is a finite union of products with factors in $\{C_1, \dots, C_m\}$.

We can suppose without loss of generality that $C_k \subset I_k := (2k, 2k+1) \subset \mathbb{R}$ for all $k \in \{1, \dots, m\}$. Thus, if $A \subset \Gamma^n$ is any product with factors in $\{C_1, \dots, C_m\} \cup \{\Gamma\}$, then A is open in Γ^n , because it can be written as $A = \Gamma^n \cap P$ for some product $P \subset \mathbb{R}^n$ with factors in $\{I_1, \dots, I_m\} \cup \{\mathbb{R}\}$.

STEP 2. Fix $E \in \text{Prox}(\tau)$. For $i \in \{0, 1\}$, consider the equivalence relation \approx_E^i on $\{1, \dots, n\}$ given by

$$p \approx_E^i q \Leftrightarrow (p, i)E(q, i).$$

Set

$$D_{\approx_E^i} := \{\gamma \in \Gamma^n : p \approx_E^i q \Rightarrow \gamma_p = \gamma_q\}$$

and define $\varphi_E^i : D_{\approx_E^i} \rightarrow \Gamma^{\ell_E}$ by

$$\varphi_E^i(\gamma)_{[(k, i)]} := \gamma_k, \quad [(k, i)] \in \ell_E, \quad \gamma \in D_{\approx_E^i}.$$

Let $R \subset \Gamma^{\ell_E}$ be any product with factors in $\{C_1, \dots, C_m\}$. It is easy to check that there is some product $A \subset \Gamma^n$ with factors in $\{C_1, \dots, C_m\} \cup \{\Gamma\}$ (in particular, A is open in Γ^n) such that $(\varphi_E^i)^{-1}(R) = D_{\approx_E^i} \cap A$, hence

$$(\varphi_E^i)^{-1}(R) \cup \Gamma^n \setminus D_{\approx_E^i} = A \cup \Gamma^n \setminus D_{\approx_E^i}.$$

Since

$$\Gamma^n \setminus D_{\approx_E^i} = \Gamma^n \cap \bigcup_{p \approx_E^i q} \{\gamma \in \mathbb{R}^n : \gamma_p \neq \gamma_q\},$$

we conclude that $(\varphi_E^i)^{-1}(R) \cup \Gamma^n \setminus D_{\approx_E^i}$ is open in Γ^n .

It follows that the sets

$$\tilde{A}_E := (\varphi_E^0)^{-1}(W_E) \cup \Gamma^n \setminus D_{\approx_E^0}$$

$$\tilde{B}_E := (\varphi_E^1)^{-1}(\Gamma^{\ell_E} \setminus W_E) \cup \Gamma^n \setminus D_{\approx_E^1}$$

are open in Γ^n . Moreover, since

$$L_E^0(S) = \varphi_E^0(S \cap D_{\approx_E^0}) \quad \text{and} \quad L_E^1(S) = \varphi_E^1(S \cap D_{\approx_E^1}) \quad \text{for every } S \subset \Gamma^n,$$

we have:

- $\varphi_E^0(\gamma) \in L_E^0(A) \subset W_E$ for every $\gamma \in A \cap D_{\approx_E^0}$, hence $A \subset \tilde{A}_E$;
- $L_E^0(\tilde{A}_E) = \varphi_E^0(\tilde{A}_E \cap D_{\approx_E^0}) \subset W_E$;
- $\varphi_E^1(\gamma) \in L_E^1(B) \subset \Gamma^{\ell_E} \setminus W_E$ for every $\gamma \in B \cap D_{\approx_E^1}$, hence $B \subset \tilde{B}_E$;
- $L_E^1(\tilde{B}_E) = \varphi_E^1(\tilde{B}_E \cap D_{\approx_E^1}) \subset \Gamma^{\ell_E} \setminus W_E$.

STEP 3. Now let

$$\tilde{A} := \bigcap_{E \in \text{Prox}(\tau)} \tilde{A}_E \quad \text{and} \quad \tilde{B} := \bigcap_{E \in \text{Prox}(\tau)} \tilde{B}_E.$$

For each $E \in \text{Prox}(\tau)$ we have

$$L_E^0(\tilde{A}) \cap L_E^1(\tilde{B}) \subset L_E^0(\tilde{A}_E) \cap L_E^1(\tilde{B}_E) \subset W_E \cap (\Gamma^{\ell_E} \setminus W_E) = \emptyset,$$

hence Lemma 2.16 ensures that \tilde{A} and \tilde{B} are τ -separated. Since \tilde{A} and \tilde{B} are open in Γ^n (bear in mind that $\text{Prox}(\tau)$ is finite), an appeal to Lemma 2.22 allows us to deduce that $\Phi(\tilde{A})$ and $\Phi(\tilde{B})$ are Σ -separated. But $A \subset \tilde{A}$ and $B \subset \tilde{B}$, so the sets $\Phi(A)$ and $\Phi(B)$ are Σ -separated as well. This finishes the proof. \square

Proof of Lemma 2.7. In view of Lemma 2.16, it suffices to prove that, for any set $\mathcal{R} \subset \text{Prox}(\tau)$, the following statement holds:

$\langle \mathcal{R} \rangle$ If $A, B \subset \Gamma^n$ satisfy:

- (i) $L_E^0(A) \cap L_E^1(B) = \emptyset$ for every $E \in \mathcal{R}$,
- (ii) for each $E \in \text{Prox}(\tau) \setminus \mathcal{R}$ there is $W_E \subset \Gamma^{\ell_E}$ in the algebra generated by products such that $L_E^0(A) \subset W_E$ and $L_E^1(B) \cap W_E = \emptyset$,

then $\Phi(A)$ and $\Phi(B)$ are Σ -separated.

We proceed by induction on $|\mathcal{R}|$. The case $|\mathcal{R}| = 0$ (i.e. $\mathcal{R} = \emptyset$) has been proved in Lemma 2.24. So assume that $|\mathcal{R}| \geq 1$ and that $\langle \mathcal{R}' \rangle$ holds true for every subset of $\text{Prox}(\tau)$ with cardinality less than $|\mathcal{R}|$. Take $A, B \subset \Gamma^n$ satisfying conditions (i) and (ii) above. We will check that $\Phi(A)$ and $\Phi(B)$ are Σ -separated.

Fix $E_0 \in \mathcal{R}$ and set $\mathcal{R}' := \mathcal{R} \setminus \{E_0\}$. For each $E \in \text{Prox}(\tau) \setminus \{E_0\}$, fix disjoint sets $X_E, Y_E \subset \Gamma^{\ell_E}$ as follows:

- $X_E := L_E^0(A)$ and $Y_E := L_E^1(B)$ for $E \in \mathcal{R}$,
- $X_E := W_E$ and $Y_E := \Gamma^{\ell_E} \setminus W_E$ for $E \in \text{Prox}(\tau) \setminus \mathcal{R}$.

Let \mathfrak{V} be as in Lemma 2.18. We claim that every $W \subset \Gamma^{\ell_{E_0}}$ in the algebra generated by products belongs to \mathfrak{V} . Indeed, let $A', B' \subset \Gamma^n$ be sets satisfying $L_E^0(A') \subset X_E$ and $L_E^1(B') \subset Y_E$ for every $E \in \text{Prox}(\tau) \setminus \{E_0\}$, $L_{E_0}^0(A') \subset W$ and $L_{E_0}^1(B') \cap W = \emptyset$. Then:

- $L_E^0(A') \cap L_E^1(B') \subset X_E \cap Y_E = \emptyset$ for every $E \in \mathcal{R}'$,
- for each $E \in \text{Prox}(\tau) \setminus \mathcal{R}'$ there is $W'_E \subset \Gamma^{\ell_E}$ in the algebra generated by products such that $L_E^0(A') \subset W'_E$ and $L_E^1(B') \cap W'_E = \emptyset$ (take $W'_{E_0} := W$ and $W'_E := W_E$ for $E \neq E_0$).

Since $\langle \mathcal{R}' \rangle$ holds, the sets $\Phi(A')$ and $\Phi(B')$ are Σ -separated. Therefore, $W \in \mathfrak{V}$.

Thus, \mathfrak{V} contains the algebra on $\Gamma^{\ell_{E_0}}$ generated by products. Since \mathfrak{V} is a monotone class (by Lemma 2.18), from the Monotone Class Theorem it follows

that the σ -algebra on $\Gamma^{\ell_{E_0}}$ generated by products is contained in \mathfrak{V} . Now, the fact that $|\Gamma|$ is a Kunen cardinal implies that $\mathfrak{V} = \mathcal{P}(\Gamma^{\ell_{E_0}})$.

In particular, the set $W := L_{E_0}^0(A)$ belongs to \mathfrak{V} . Since $L_E^0(A) \subset X_E$ and $L_E^1(B) \subset Y_E$ for every $E \in \text{Prox}(\tau) \setminus \{E_0\}$, $L_{E_0}^0(A) \subset W$ and $L_{E_0}^1(B) \cap W = \emptyset$, we conclude that $\Phi(A)$ and $\Phi(B)$ are Σ -separated. This proves that $\langle \mathcal{R} \rangle$ holds and the proof of Lemma 2.7 is over. \square

3. THE CASE OF $C(2^{\omega_1})$

The aim of this section is to give a different, more direct proof of the equality $\text{Ba}(C_p(2^{\omega_1})) = \text{Bo}(C(2^{\omega_1}))$, see Theorem 3.6 below.

We denote by \mathfrak{G} the family of all open intervals of \mathbb{R} with rational endpoints and we write $\mathcal{J} := \bigcup_{n \in \mathbb{N}} \mathfrak{G}^n$. Given a compact space K , $n \in \mathbb{N}$, $A \subset K^n$ and $J = (J_1, \dots, J_n) \in \mathfrak{G}^n$, we define

$$u(A, J) := \{g \in C(K) : \text{there is } (x_1, \dots, x_n) \in A \text{ such that } g(x_k) \in J_k \text{ for all } k = 1, \dots, n\}.$$

Remark 3.1. In the previous conditions, we have $u(A, J) = u(\overline{A}, J)$.

Proof. For any $g \in C(K)$, the set $U := \prod_{k=1}^n g^{-1}(J_k) \subset K^n$ is open, and therefore $U \cap \overline{A} \neq \emptyset$ if and only if $U \cap A \neq \emptyset$. \square

In Corollary 3.4 we shall isolate a property of a compact space K guaranteeing that $\text{Ba}(C_p(K)) = \text{Bo}(C_p(K))$. To this end we need a couple of lemmas.

Lemma 3.2. *Let K be a compact space such that $u(F, J) \in \text{Ba}(C_p(K))$ for every closed set $F \subset K^n$, every $J \in \mathfrak{G}^n$ and every $n \in \mathbb{N}$. Then $\text{Ba}(C_p(K)) = \text{Bo}(C_p(K))$.*

Proof. Let $G \subset C(K)$ be open for the pointwise convergence topology. For $n \in \mathbb{N}$ and $J = (J_1, \dots, J_n) \in \mathfrak{G}^n$, set $A_J := \bigcup \{A \subset K^n : u(A, J) \subset G\}$, so that $u(A_J, J) \subset G$. We claim that

$$(3.1) \quad G = \bigcup_{J \in \mathcal{J}} u(A_J, J).$$

Indeed, given any $g \in G$, we can find $\{t_1, \dots, t_n\} \subset K$ and $J = (J_1, \dots, J_n) \in \mathfrak{G}^n$ such that

$$g \in H := \{h \in C(K) : h(t_k) \in J_k \text{ for all } k = 1, \dots, n\} \subset G.$$

Since $u(\{(t_1, \dots, t_n)\}, J) = H \subset G$, we have $(t_1, \dots, t_n) \in A_J$ and so $g \in u(A_J, J)$. This proves equality (3.1). Now, in view of Remark 3.1, we get

$$G = \bigcup_{J \in \mathcal{J}} u(\overline{A_J}, J).$$

Since \mathcal{J} is countable and each $u(\overline{A_J}, J)$ belongs to $\text{Ba}(C_p(K))$ (by the assumption), it follows that $G \in \text{Ba}(C_p(K))$. Hence $\text{Ba}(C_p(K)) = \text{Bo}(C_p(K))$. \square

Lemma 3.3. *Let K be a compact space, $n \in \mathbb{N}$, $J \in \mathfrak{G}^n$ and $(F_p)_{p \in \mathbb{N}}$ a decreasing sequence of closed separable subsets of K^n . Then $u(\bigcap_{p \in \mathbb{N}} F_p, J) \in \text{Ba}(C_p(K))$.*

Proof. We divide the proof into two steps.

STEP 1. $u(S, J) \in \text{Ba}(C_p(K))$ for every closed separable set $S \subset K^n$. Indeed, take $D \subset S$ countable with $\overline{D} = S$. By Remark 3.1, we have

$$u(S, J) = u(D, J) = \bigcup_{x \in D} u(\{x\}, J).$$

Since each $u(\{x\}, J)$ belongs to $\text{Ba}(C_p(K))$, the same holds for $u(S, J)$.

STEP 2. Write $J = (J_1, \dots, J_n)$ and set $F := \bigcap_{p \in \mathbb{N}} F_p$. For each $m \in \mathbb{N}$, choose

$$J^m = (J_1^m, \dots, J_n^m) \in \mathfrak{G}^n$$

such that $\overline{J_k^m} \subset J_k^{m+1}$ and $\bigcup_{m \in \mathbb{N}} J_k^m = J_k$ for every $m \in \mathbb{N}$ and $k \in \{1, \dots, n\}$. According to Step 1, in order to prove that $u(F, J) \in \text{Ba}(C_p(K))$ it suffices to check that

$$(3.2) \quad u(F, J) = \bigcup_{m \in \mathbb{N}} \bigcap_{p \in \mathbb{N}} u(F_p, J^m).$$

To this end, observe first that if $g \in u(F, J)$ then there is $(x_1, \dots, x_n) \in F$ such that $g(x_k) \in J_k$ for all k . Since $J_k = \bigcup_{m \in \mathbb{N}} J_k^m$ and $J_k^m \subset J_k^{m+1}$, we can find $m \in \mathbb{N}$ large enough such that $g(x_k) \in J_k^m$ for all k , hence $g \in u(F, J^m) \subset \bigcap_{p \in \mathbb{N}} u(F_p, J^m)$.

To check “ \supset ” in (3.2), fix $g \in \bigcup_{m \in \mathbb{N}} \bigcap_{p \in \mathbb{N}} u(F_p, J^m)$. Then there exists $m \in \mathbb{N}$ such that, for each $p \in \mathbb{N}$, there is some $x^p = (x_1^p, \dots, x_n^p) \in F_p$ with the property that $g(x_k^p) \in J_k^m$ for all k . Let $x \in K^n$ be any cluster point of the sequence $(x^p)_{p \in \mathbb{N}}$. Then $x \in F$ and $g(x_k) \in \overline{J_k^m} \subset J_k$ for all k , witnessing that $g \in u(F, J)$. This proves (3.2) and we are done. \square

As an immediate consequence of Lemmas 3.2 and 3.3 we get:

Corollary 3.4. *Let K be a compact space such that, for each $n \in \mathbb{N}$ and each closed set $F \subset K^n$, there is a decreasing sequence $(F_p)_{p \in \mathbb{N}}$ of closed separable subsets of K^n such that $F = \bigcap_{p \in \mathbb{N}} F_p$. Then $\text{Ba}(C_p(K)) = \text{Bo}(C_p(K))$.*

It turns out that the previous criterion can be applied to 2^{ω_1} , as we next show.

Lemma 3.5. *For each closed set $F \subset 2^{\omega_1}$ there is a decreasing sequence $(F_p)_{p \in \mathbb{N}}$ of closed separable subsets of 2^{ω_1} such that $F = \bigcap_{p \in \mathbb{N}} F_p$.*

Proof. By Parovichenko’s theorem (cf. [11, 3.12.18]), every compact space of weight less than or equal to ω_1 (like F) is a continuous image of $\beta\mathbb{N} \setminus \mathbb{N}$. Let $q : \beta\mathbb{N} \setminus \mathbb{N} \rightarrow 2^{\omega_1}$ be a continuous mapping with $q(\beta\mathbb{N} \setminus \mathbb{N}) = F$. Then q can be extended to a continuous mapping $g : \beta\mathbb{N} \rightarrow 2^{\omega_1}$. Indeed, fix $\alpha < \omega_1$, let $\pi_\alpha : 2^{\omega_1} \rightarrow \{0, 1\}$ be the α -th coordinate projection and apply Tietze’s theorem to find a continuous mapping $f_\alpha : \beta\mathbb{N} \rightarrow [0, 1]$ such that $f_\alpha|_{\beta\mathbb{N} \setminus \mathbb{N}} = \pi_\alpha \circ q$. Since $f_\alpha^{-1}(\{0\})$ and $f_\alpha^{-1}(\{1\})$ are disjoint closed subsets of the 0-dimensional compact space $\beta\mathbb{N}$, there is a clopen set $A_\alpha \subset \beta\mathbb{N}$ such that $f_\alpha^{-1}(\{0\}) \cap A_\alpha = \emptyset$ and $f_\alpha^{-1}(\{1\}) \subset A_\alpha$. Now, it is easy to check that the continuous mapping $g : \beta\mathbb{N} \rightarrow 2^{\omega_1}$ defined by $\pi_\alpha \circ g := 1_{A_\alpha}$ for all $\alpha < \omega_1$ satisfies $g|_{\beta\mathbb{N} \setminus \mathbb{N}} = q$.

For each $p \in \mathbb{N}$, the set $Z_p := \beta\mathbb{N} \setminus \{1, \dots, p\}$ is closed and separable, hence the same holds for $F_p := g(Z_p) \subset 2^{\omega_1}$. Since $(Z_p)_{p \in \mathbb{N}}$ is a decreasing sequence of compact sets and g is continuous, we have

$$\bigcap_{p \in \mathbb{N}} F_p = \bigcap_{p \in \mathbb{N}} g(Z_p) = g\left(\bigcap_{p \in \mathbb{N}} Z_p\right) = g(\beta\mathbb{N} \setminus \mathbb{N}) = q(\beta\mathbb{N} \setminus \mathbb{N}) = F,$$

and the proof is over. \square

Finally, we can give an alternative proof of the following:

Theorem 3.6. $\text{Ba}(C_p(2^{\omega_1})) = \text{Bo}(C(2^{\omega_1}))$.

Proof. As we pointed out in the introduction, for any cardinal κ we always have

$$\text{Bo}(C_p(2^\kappa)) = \text{Bo}(C(2^\kappa)).$$

On the other hand, $\text{Ba}(C_p(2^{\omega_1})) = \text{Bo}(C_p(2^{\omega_1}))$, by Corollary 3.4 and Lemma 3.5 (bear in mind that all finite powers of 2^{ω_1} are homeomorphic to 2^{ω_1}). \square

Remark 3.7. Let us say that κ is a *Parovicenko cardinal* if every compact space of weight less than or equal to κ is a continuous image of $\beta\mathbb{N} \setminus \mathbb{N}$. This is the only property of the cardinal ω_1 that we have used in the proofs of Lemma 3.5 and Theorem 3.6, so we have indeed shown that:

$$\text{Ba}(C_p(2^\kappa)) = \text{Bo}(C(2^\kappa)) \text{ whenever } \kappa \text{ is a Parovicenko cardinal.}$$

Notice that van Douwen and Przymusiński [6] proved that, under Martin's axiom, all cardinals $< \mathfrak{c}$ are Parovicenko cardinals. We do not know whether the analogue of Lemma 3.5 for 2^κ is true if κ is a Kunen cardinal.

Recall that a Banach space X is *measure-compact* (in its weak topology) if and only if, for each probability measure μ on $\text{Ba}(X_w)$, there is a separable subspace X_0 of X such that $\mu^*(X_0) = 1$. Such a property has been considered in connection with Pettis integration, see e.g. [10, 28]. The following consequence of Theorem 3.6 was first proved in [23] by a completely different approach.

Corollary 3.8. $C(2^{\omega_1})$ is measure-compact.

Proof. Let μ be a probability measure on $\text{Ba}(C_w(2^{\omega_1})) = \text{Bo}(C(2^{\omega_1}))$. Since the metric space $C(2^{\omega_1})$ has density character ω_1 (which is not real-valued measurable), a classical result due to Marczewski and Sikorski (cf. [20, Theorem III]) ensures that μ has a separable support. \square

In Corollary 3.8 one can replace ω_1 by any κ which is a Kunen cardinal, since in such a case no cardinal $\kappa_1 \leq \kappa$ is real-valued measurable, see [17]. However, for $\kappa > \omega_1$ the result of [23] is more general: under the absence of weakly inaccessible cardinals $C(2^\kappa)$ is measure-compact for every κ .

Let us also mention another consequence of Theorem 3.6; cf. [22] for some results on Borel structures in nonseparable metric spaces. We refer to [5] for the definition of cardinal \mathfrak{p} .

Corollary 3.9 ($\mathfrak{p} > \omega_1$). $\text{Bo}(C(2^{\omega_1}))$ is countably generated.

Proof. Let $A \subset 2^{\omega_1}$ be a countable dense set and let Σ be the σ -algebra on $C(2^{\omega_1})$ generated by $\{\delta_a : a \in A\}$. Clearly, Σ is countably generated. It follows from $\mathfrak{p} > \omega_1$ that every $x \in 2^{\omega_1}$ is a limit of a converging sequence from A , see e.g. [5, Theorem 6.2]. This implies that δ_x is Σ -measurable for every $x \in 2^{\omega_1}$, and we get $\Sigma = \text{Ba}(C_p(2^{\omega_1})) = \text{Bo}(C(2^{\omega_1}))$, which completes the proof. \square

4. NON WEAK BAIRE MEASURABLE NORMS

An equivalent norm on a Banach space X is $\text{Ba}(X_w)$ -measurable (as a real-valued function defined on X) if and only if its balls belong to $\text{Ba}(X_w)$. Clearly, this implies that all singletons belong to $\text{Ba}(X_w)$, which is equivalent to saying that the dual X^* is w^* -separable, cf. [16, Theorem 1.5.3]. There are Banach spaces with w^* -separable dual which admit a non $\text{Ba}(X_w)$ -measurable equivalent norm, like ℓ^∞ and the Johnson-Lindenstrauss spaces, see [24]. Obviously, if the equality $\text{Ba}(X_w) = \text{Bo}(X)$ holds, then all equivalent norms on X are $\text{Ba}(X_w)$ -measurable. The aim of this section is to show that the converse holds for $C(2^\kappa)$ and $\ell^1(\kappa)$, see Corollary 4.5.

Recall that a function $f : \Omega \rightarrow X$ from a measurable space (Ω, Σ) to a Banach space X is called *scalarly measurable* if the composition $x^* \circ f$ is Σ -measurable for every $x^* \in X^*$, i.e. f is Σ - $\text{Ba}(X_w)$ -measurable. We shall also use the following notion introduced in [14]:

Definition 4.1. Let X be a Banach space. A family $\{(x_\alpha, x_\alpha^*) : \alpha \in I\} \subset X \times X^*$ is called a bounded almost biorthogonal system (BABS) of type $\eta \in [0, 1)$ if

- (i) $\{x_\alpha : \alpha \in I\}$ and $\{x_\alpha^* : \alpha \in I\}$ are bounded,
- (ii) $x_\alpha^*(x_\alpha) = 1$ for every $\alpha \in I$,
- (iii) $|x_\alpha^*(x_\beta)| \leq \eta$ whenever $\alpha \neq \beta$.

Lemma 4.2. Let X be a Banach space having a BABS $\{(x_\alpha, x_\alpha^*) : \alpha \in I\}$ of type $\eta \in [0, 1)$. Suppose there is a measurable space (Ω, Σ) and a mapping $i : \Omega \rightarrow I$ such that:

- the function $f : \Omega \rightarrow X$ defined by $f(\theta) := x_{i(\theta)}$ is scalarly measurable,
- there is $A \subset I$ such that $i^{-1}(A) \notin \Sigma$.

Then there is an equivalent norm on X which is not $\text{Ba}(X_w)$ -measurable.

Proof. Fix an equivalent norm $\|\cdot\|$ on X and set $C := \sup\{\|x_\alpha\| : \alpha \in I\}$. The formula

$$\|x\|_0 := C^{-1} \max \left\{ \|x\|, C \sup_{\alpha \in I} |x_\alpha^*(x)| \right\}$$

defines an equivalent norm on X (bear in mind that $\{x_\alpha^* : \alpha \in I\}$ is bounded) such that $\|x_\alpha\|_0 = 1$ for all $\alpha \in I$. Fix $1 < u < v < \eta^{-1}$ (with the convention $0^{-1} = \infty$) and set $b(\alpha) := u$ if $\alpha \in A$, $b(\alpha) := v$ if $\alpha \in I \setminus A$. The formula

$$|x| := \max \left\{ \|x\|_0, \sup_{\alpha \in I} b(\alpha) |x_\alpha^*(x)| \right\}$$

defines another equivalent norm on X .

We claim that $|\cdot|$ is not $\text{Ba}(X_w)$ -measurable. To prove this, it suffices to check that the real-valued function $\theta \mapsto |f(\theta)|$ is not Σ -measurable (bear in mind that f is $\Sigma\text{-Ba}(X_w)$ -measurable). Fix $\theta \in \Omega$. For each $\alpha \in I$ with $\alpha \neq i(\theta)$ we have $|x_\alpha^*(f(\theta))| = |x_\alpha^*(x_{i(\theta)})| \leq \eta$ and so

$$b(\alpha)|x_\alpha^*(f(\theta))| \leq b(\alpha)\eta < 1 = \|f(\theta)\|_0.$$

On the other hand, $b(i(\theta))|x_{i(\theta)}^*(f(\theta))| = b(i(\theta)) > 1 = \|f(\theta)\|_0$. It follows that

$$\begin{aligned} |f(\theta)| &= \max \left\{ \|f(\theta)\|_0, \sup_{\alpha \in I} b(\alpha)|x_\alpha^*(f(\theta))| \right\} = \\ &= b(i(\theta)) = u1_{i^{-1}(A)}(\theta) + v1_{\Omega \setminus i^{-1}(A)}(\theta) \end{aligned}$$

for all $\theta \in \Omega$. Since $i^{-1}(A) \notin \Sigma$, the function $\theta \mapsto |f(\theta)|$ is not Σ -measurable. \square

Lemma 4.3. *Let X be a Banach space having a bounded biorthogonal system $\{(x_\alpha, x_\alpha^*) : \alpha \in I\}$. Let $U \subset I \times I$ be a set such that:*

- (a) $\alpha \neq \beta$ for every $(\alpha, \beta) \in U$,
- (b) $(\beta, \alpha) \notin U$ whenever $(\alpha, \beta) \in U$.

Then:

- (i) The family

$$(4.1) \quad \left\{ \left(x_\alpha + x_\beta, \frac{x_\alpha^* + x_\beta^*}{2} \right) : (\alpha, \beta) \in U \right\} \subset X \times X^*$$

is a BABS of type 1/2.

- (ii) The function $f : U \rightarrow X$ given by $f(\alpha, \beta) := x_\alpha + x_\beta$ is scalarly measurable when U is equipped with the trace of $\mathcal{P}(I) \otimes \mathcal{P}(I)$.

Proof. To prove (i), fix (α, β) and (α', β') in U . Then

$$d := (x_\alpha^* + x_\beta^*)(x_{\alpha'} + x_{\beta'}) = \delta_{\alpha, \alpha'} + \delta_{\alpha, \beta'} + \delta_{\beta, \alpha'} + \delta_{\beta, \beta'}$$

and therefore:

- If $(\alpha, \beta) = (\alpha', \beta')$, then $\alpha \neq \beta'$ and $\alpha' \neq \beta$ (by (a)), hence $d = 2$.
- If $\alpha = \alpha'$ and $\beta \neq \beta'$, then $\alpha \neq \beta'$ and $\alpha' \neq \beta$ (by (a)), hence $d = 1$.
- If $\alpha \neq \alpha'$ and $\beta = \beta'$, then $\alpha \neq \beta'$ and $\alpha' \neq \beta$ (by (a)), hence $d = 1$.
- If $\alpha \neq \alpha'$ and $\beta \neq \beta'$, then $d \in \{0, 1\}$, because in this case we have $\alpha \neq \beta'$ whenever $\alpha' = \beta$ (by (b)).

It follows that (4.1) is a BABS of type 1/2.

To prove (ii), fix $x^* \in X^*$. For each $r \in \mathbb{R}$, the set

$$\begin{aligned} \{(\alpha, \beta) \in U : x^*f(\alpha, \beta) < r\} &= \{(\alpha, \beta) \in U : x^*(x_\alpha) + x^*(x_\beta) < r\} = \\ &= \bigcup_{\substack{p, q \in \mathbb{Q} \\ p+q < r}} \{(\alpha, \beta) \in U : x^*(x_\alpha) < p, x^*(x_\beta) < q\} = \\ &= U \cap \bigcup_{\substack{p, q \in \mathbb{Q} \\ p+q < r}} \{\alpha \in I : x^*(x_\alpha) < p\} \times \{\beta \in I : x^*(x_\beta) < q\} \end{aligned}$$

belongs to the trace of $\mathcal{P}(I) \otimes \mathcal{P}(I)$ on U . So, f is scalarly measurable. \square

We arrive at the key result of this section.

Theorem 4.4. *Let X be a Banach space having a biorthogonal system of non Kunen cardinality. Then there exists an equivalent norm on X which is not $\text{Ba}(X_w)$ -measurable.*

Proof. Let κ be a non Kunen cardinal such that X has a biorthogonal system of cardinality κ . Suppose first that $\kappa > \mathfrak{c}$. Then $|X| > \mathfrak{c}$ and so X^* is not w^* -separable (bear in mind that any Banach space having w^* -separable dual injects into ℓ^∞). Thus, in this case *all* equivalent norms on X are not $\text{Ba}(X_w)$ -measurable.

Suppose now that $\kappa \leq \mathfrak{c}$. Fix a *bounded* biorthogonal system

$$\{(x_\alpha, x_\alpha^*) : \alpha \in I\} \subset X \times X^*$$

with $|I| = \kappa$ (cf. [15, Theorem 4.15]). We can assume that $I \subset \mathbb{R}$. Then

$$U := \{(\alpha, \beta) \in I \times I : \alpha > \beta\} \quad \text{and} \quad V := \{(\alpha, \beta) \in I \times I : \alpha < \beta\}$$

belong to $\mathcal{P}(I) \otimes \mathcal{P}(I)$, because they can be written as

$$U = \bigcup_{\substack{p, q \in \mathbb{Q} \\ p > q}} I \cap (p, \infty) \times I \cap (-\infty, q) \quad \text{and} \quad V = \bigcup_{\substack{p, q \in \mathbb{Q} \\ p < q}} I \cap (-\infty, p) \times I \cap (q, \infty).$$

Since $|I|$ is not a Kunen cardinal, there is a set $B \subset I \times I$ which does not belong to $\mathcal{P}(I) \otimes \mathcal{P}(I)$. As we noticed in the proof of Lemma 2.2, we have

$$B \setminus (U \cup V) \in \mathcal{P}(I) \otimes \mathcal{P}(I),$$

therefore either $B \cap U \notin \mathcal{P}(I) \otimes \mathcal{P}(I)$ or $B \cap V \notin \mathcal{P}(I) \otimes \mathcal{P}(I)$. From now on we assume that $B \cap U \notin \mathcal{P}(I) \otimes \mathcal{P}(I)$ (the other case is analogous).

Let Σ_U be the trace σ -algebra of $\mathcal{P}(I) \otimes \mathcal{P}(I)$ on U . Observe that U satisfies conditions (a) and (b) of Lemma 4.3, hence the family

$$\left\{ \left(x_\alpha + x_\beta, \frac{x_\alpha^* + x_\beta^*}{2} \right) : (\alpha, \beta) \in U \right\} \subset X \times X^*$$

is a BABS of type 1/2 and the function $f : U \rightarrow X$ given by $f(\alpha, \beta) := x_\alpha + x_\beta$ is scalarly measurable with respect to Σ_U . Since $A := B \cap U \notin \Sigma_U$ (bear in mind that $\Sigma_U \subset \mathcal{P}(I) \otimes \mathcal{P}(I)$), an appeal to Lemma 4.2 ensures the existence of a non $\text{Ba}(X_w)$ -measurable equivalent norm on X . The proof is over. \square

Let κ be a cardinal. For each $\alpha < \kappa$, define $(e_\alpha, e_\alpha^*) \in \ell^1(\kappa) \times \ell^1(\kappa)^*$ by declaring $e_\alpha(\beta) := \delta_{\alpha, \beta}$ for all $\beta < \kappa$ and $e_\alpha^*(f) := f(\alpha)$ for all $f \in \ell^1(\kappa)$. Then $\{(e_\alpha, e_\alpha^*) : \alpha < \kappa\}$ is a biorthogonal system. Moreover, since $\ell^1(\kappa)$ is isomorphic to a closed subspace of $C(2^\kappa)$, the Hahn-Banach theorem ensures that $C(2^\kappa)$ also has a biorthogonal system of cardinality κ . From Theorems 2.8 and 4.4 we now get:

Corollary 4.5. *The following statements are equivalent for a cardinal κ :*

- (i) κ is a Kunen cardinal.
- (ii) All equivalent norms on $\ell^1(\kappa)$ are $\text{Ba}(\ell^1(\kappa)_w)$ -measurable.
- (iii) All equivalent norms on $C(2^\kappa)$ are $\text{Ba}(C_w(2^\kappa))$ -measurable.

It is clear that an equivalent norm on a Banach space X is $\text{Ba}(X_w)$ -measurable whenever its closed dual unit ball is w^* -separable. However, the converse is not true in general (for an example with $X = \ell^\infty$, see [24]). On the other hand, it was shown in [14] that the following properties are equivalent:

- (i) All equivalent norms on X have w^* -separable closed dual unit ball.
- (ii) There is no *uncountable* BABS on X .

Moreover, when X is a dual space, (i) and (ii) are equivalent to the separability of X , cf. [15, Corollary 4.34]. Our last result complements such equivalence.

Proposition 4.6. *Let Y be a separable Banach space not containing ℓ^1 . The following statements are equivalent:*

- (i) Y^* is separable.
- (ii) All equivalent norms on Y^* are $\text{Ba}(Y_w^*)$ -measurable.

Proof. It only remains to prove (ii) \Rightarrow (i). Since Y is separable, its dual $X := Y^*$ is a representable Banach space. Thus, if we assume that X is not separable, then there is a bounded biorthogonal system $\{(x_\alpha, x_\alpha^*) : \alpha < \mathfrak{c}\} \subset X \times X^*$, cf. [15, Theorem 4.33]. Let $D \subset Y$ be a countable norm dense set. We claim that

$$(4.2) \quad \text{Ba}(X_w) = \sigma(D).$$

Indeed, fix $y^{**} \in X^* = Y^{**}$. By the Odell-Rosenthal theorem (cf. [7, Theorem 4.1]) there is a sequence $(y_n)_{n \in \mathbb{N}}$ in Y converging to y^{**} in the w^* -topology. Since D is norm dense in Y , we can find $y'_n \in D$ such that $\|y_n - y'_n\| \leq 1/n$. Then $(y'_n)_{n \in \mathbb{N}}$ also converges to y^{**} in the w^* -topology and so y^{**} is $\sigma(D)$ -measurable. As $y^{**} \in X^*$ is arbitrary, equality (4.2) holds.

In particular, $\text{Ba}(X_w)$ is countably generated. Thus, $|\text{Ba}(X_w)| = \mathfrak{c} < 2^{\mathfrak{c}}$ and hence there exists $A \subset \mathfrak{c}$ such that $\{x_\alpha : \alpha \in A\}$ does not belong to the trace of $\text{Ba}(X_w)$ on $\Omega := \{x_\alpha : \alpha < \mathfrak{c}\}$, which we denote by Σ . Since the “identity” function $f : \Omega \rightarrow X$ satisfies the assumptions of Lemma 4.2 (with respect to Σ), the space X admits a non $\text{Ba}(X_w)$ -measurable equivalent norm. \square

Remark 4.7. If \mathfrak{c} is not a Kunen cardinal, then statements (i) and (ii) of Proposition 4.6 are equivalent for any separable Banach space Y .

Proof. It only remains to prove that (ii) fails when Y contains ℓ^1 . In this case, $\ell^1(\mathfrak{c})$ is isomorphic to a closed subspace Z of Y^* (cf. [7, Theorem 4.1]). By Corollary 4.5, there is a non $\text{Ba}(Z_w)$ -measurable equivalent norm $\|\cdot\|_Z$ on Z . Since the trace of $\text{Ba}(Y_w^*)$ on Z is exactly $\text{Ba}(Z_w)$, we conclude that any equivalent norm on Y^* extending $\|\cdot\|_Z$ (cf. [4, II.8.1]) cannot be $\text{Ba}(Y_w^*)$ -measurable. \square

However, if \mathfrak{c} is a Kunen cardinal, then $\text{Ba}(C[0, 1]_w^*) = \text{Bo}(C[0, 1]^*)$ (see Remark 2.12) and so all equivalent norms on $C[0, 1]^*$ are $\text{Ba}(C[0, 1]_w^*)$ -measurable, while $C[0, 1]^*$ is nonseparable.

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